The Mathematics

Suppose you wish to borrow a dollar amount A and make N payments of P Dollars each, at an interest rate of I percent per payment period, leaving you with a future value of F Dollars. This page describes the mathematics connecting these five quantities.

Let a_n denote the value of your account after n payment periods. Thus

$$a_0 = A \quad \text{and} \quad a_N = F. \tag{1}$$

The key equation for the whole analysis describes the transition from one payment period to the next:

$$a_{n+1} = a_n + \frac{I}{100} a_n - P. \tag{2}$$

In the very special case that $I = 0$ the equation (2) turns into

$$a_{n+1} = a_n - P \tag{3}$$

and we get

$$F = A - NP, \tag{4}$$

$$A = F + NP, \tag{5}$$

$$N = \frac{A - F}{P}, \quad \text{and} \tag{6}$$

$$P = \frac{A - F}{N}. \tag{7}$$

In the more typical (and much more complicated) case that $I \neq 0$ we rewrite the equation (2) as

$$a_{n+1} = \eta a_n - P \quad \text{where} \quad \eta = 1 - \frac{I}{100}. \tag{8}$$

Specifically we obtain:

$$a_0 = A \tag{9}$$

$$a_1 = \eta A - P \tag{10}$$

$$a_2 = \eta^2 A - (1 + \eta) P \tag{11}$$

$$a_3 = \eta^3 A - (1 + \eta + \eta^2) P \tag{12}$$

$$\ldots = \ldots \tag{13}$$

$$a_n = \eta^n A - \sum_{k=0}^{n-1} \eta^n P, \quad n = 0, 1, 2, \ldots \tag{14}$$

$$\tag{15}$$

Using the formula

$$\sum_{k=0}^{n-1} \eta^k = \frac{1-\eta^n}{1-\eta} \tag{16}$$

for geometric series, we can express the future value as

$$F = a_N = \eta^N A - \frac{1-\eta^N}{1-\eta} P. \tag{17}$$

This equation can be solved explicitly for

$$N = \frac{\log\left(\frac{(1-\eta)F+P}{(1-\eta)A+P}\right)}{\log(\eta)}, \tag{18}$$

$$A = \frac{F + \frac{1-\eta^N}{1-\eta}}{\eta^N}, \quad \text{and} \tag{19}$$

$$P = \frac{1-\eta}{1-\eta^N}(F - \eta^N A) \tag{20}$$

The computation of I is more complicated. The calculator attempts to solve the equation

$$Z(I) = F - \left(\eta^N A - \frac{1-\eta^N}{1-\eta} P\right) = 0 \tag{21}$$

numerically and approximately by the method of bisection. Once η is known I can be computed by the formula

$$I = 100(1-\eta). \tag{22}$$

Bisection requires an interval $[a, b]$ where $Z(a)$ and $Z(b)$ have opposite signs. To that end the calculator starts with $a = b = 0.2$ if I is positive and $a = b = -0.2$ if I is negative. It then keeps dividing a by 1.1 and multiplying b with 1.1 until $Z(a)$ and $Z(b)$ have opposite signs. That usually works but it is not fail safe.